PMM U.S.S.R.,Vol.44,pp.697-702 0021-8928/81/6 0697 $7.50/0

UDC 531.36

STABILITY OF A SET OF PROCESSES WITH AFTEREFFECT*

T. K. SIRAZETDINOV

Axiomatic description of processes with aftereffect is introduced for systems with distributed parameters. Concepts of stability of a set or pipe of processes with aftereffect are defined, and the necessary and sufficient conditions of stability and instability, which are developments of results in /1-3/, are obtained.

In the application of the method of Liapunov functions to the analysis of stability of solutions of differential equations not all properties of solutions are used. Hence it is of interest to separate only those general properties of motions or solutions of differential equations that are used in proving theorems, in constructing abstract axiomatic processes, and obtaining stability conditions for the latter.

The problem of investigation of stability or other properties of motion reduces to the test of existence of Liapunov functions that satisfy the conditions of respective theorems and of the fulfillment of a given system of axioms which include the fundamental properties of solutions of the Cauchy problem of a wide class of differential equations and many other processes.

The axiomatic description of processes was also considered in /4-6/ and other works.

1. Let (α_0, T) be an interval of the real axis where $\alpha_0 \leqslant 0$ and $T > 0$. Let us consider the set Φ_0 of elements φ_0. If to every specific time t in the interval $(t_0, t_1] \subset (\alpha_0, T)$ corresponds in Φ_0 a particular element $\varphi_0 = \varphi_0(t_0, t_1; t)$, we shall assume that the initial curve $\varphi_0 = \varphi_0(t_0, t_1; t)$ is specified in the interval $(t_0, t_1]$. It is significant that φ_0 depends not only on t but, also, on the interval $(t_0, t_1]$. For instance, when Φ_0 is a set of numbers, φ_0 assumes numerical values and the curve defined by formula $\varphi = t^2(t_1 - t_0)$ depends on the selection of the interval (t_0, t_1). For each pair of t_0, t_1 there exists in this case a specific dependence of φ_0 on time t. Function $\varphi = t^2$ which is independent of the interval $(t_0, t_1]$ is also a curve in the considered here sense.

From the multiplicity of initial curves we separate the class of curves called initial processes.

Axioms of initial processes. 1.1. Any initial process determinate in the interval $(t_0, t_1] \subset (\alpha_0, T)$ is the initial process in any interval $(t_0', t_1'] \subset (t_0, t_1]$.

1.2. If two initial processes $\varphi_0(t_0, t_1; t)$ and $\varphi_0(t_1, t_2, t)$, where $\alpha_0 \leqslant t_0 \leqslant t_1 \leqslant t_2 \leqslant T$, are specified, then the composite initial process $\varphi_0(t_0, t_2; t)$, consisting at $t \in (t_0, t_1]$ of elements of the first, and at $t \in (t_1, t_2]$ of those of the second initial process, is also an initial process.

1.3. At least one initial process determinate throughout the interval (α_0, T) exists.

We shall call $\varphi_0 \in \Phi_0$ the initial state and the three above axioms, respectively, the contraction, articulation, and existence axioms.

Axioms 1.1 and 1.3 imply that an initial process exists in any interval $(t_0, t_1] \subset (t_0, T]$. According to axiom 1.2 we obtain the initial process by combining initial processes of adjoining time intervals.

2. Let Φ be a set of elements φ. If to each time interval $t \in [t_0, t_1) \subset (0, T)$ and initial process $\varphi_0 = \varphi_0(\alpha, t_0; t')$ determinate in the interval $(\alpha, t_0] \subset (\alpha_0, t_0]$ and $t' \in (\alpha, t_0]$, a specific point $\varphi = \varphi(\varphi_0(\alpha, t_0; t'), t_0, t_1; t)$ corresponds in Φ, a curve with aftereffect is specified in the set Φ. Thus a curve with aftereffect satisfies the initial condition, i.e. the axiom of initial data.

Here the element $\varphi \in \Phi$ depends at the instant of time t on the initial process $\varphi_0 = \varphi_0(\alpha, t_0; t')$ specified in the interval $(\alpha_0, t_0] \subset (a_0, T]$ and on the interval $[t_0, t_1)$. The initial process $\varphi_0 = \varphi_0(\alpha, t_0; t')$ is also called the initial condition.

If a curve with aftereffect begins to develop in the interval $[t_0, t_1) \subset (\alpha_0, T)$, at instant of time t_0, its determination requires the knowledge of the initial process or the initial condition in the interval $(\alpha_0, t_0]$. If, however, a curve with aftereffect is considered in the interval $[t_0', t_1) \subset (\alpha_0, T)$, where $t_0 < t_0'$, it is necessary to know the initial state curve in the interval $(\alpha', t_0'] \subset (\alpha_0, t_0']$.

---

*Prikl.Matem.Mekhan.,44,No.6,977-985,1980

By specifying various initial processes in the interval $(\alpha', t_0']$, we generally obtain different curves with aftereffect in the interval $(t_0', t_1]$. If, however, the initial process $\varphi_0(\alpha', t_0'; t)$ coincides with some curve φ' with aftereffect in the interval $[t_0, t_0') \subset (\alpha_0, T)$, where $t_0 < t_0'$, the curve that has such initial process in the interval $[t_0', t_1)$ is considered as the continuation of curve φ' with aftereffect in the interval $[t_0', t)$.

From among all possible curves with aftereffect we separate a class of processes with aftereffect.

Axioms of processes with aftereffect. 2.1. Any process with aftereffect determinate in the interval $[t_0, t_1) \subset (0, T)$ is also a process with aftereffect when considered in any interval $[t_0', t_1') \subset [t_0, t_1)$, i.e. when $\varphi(\varphi_0(\alpha, t_0; t'), t_0, t_1; t)$ is a process with aftereffect, $\varphi(\varphi_0'(\alpha, t_0'; t'), t_0', t_1'; t)$ is also a process with aftereffect with initial condition

$$t' \in [\alpha, t_0), \ \varphi_0'(\alpha, t_0'; t') = \varphi_0(\alpha, t_0; t'), \quad t' \in [t_0, t_0'], \ \varphi_0'(\alpha, t_0'; t') = \varphi(\varphi_0(\alpha, t_0; t''), t_0, t_1; t')$$

2.2. If two processes

$$\varphi(\varphi_0(\alpha_0, t_0; t'), t_0, t_1; t), \ \varphi(\varphi_0(\alpha_1, t_1; t''), t_1, t_2; t)$$

with aftereffect determinate, respectively, in the intervals $[t_0, t_1)$ and $[t_1, t_2)$ are such that

$$t \in (\alpha_1, t_0], \ \alpha_0 \leqslant \alpha_1 \leqslant t_0, \ \varphi_0(\alpha_1, t_1; t) = \varphi_0(\alpha_0, t_0; t), \quad t \in [t_0, t_1], \ \varphi_0(\alpha_1, t_1; t) = \varphi(\varphi_0(\alpha_0, t_0; t'), t_0, t_1; t)$$

the composite curve consisting at $t < t_1$ of elements of one process with aftereffect and at $t \geqslant t_1$ of elements of another process is also a process with aftereffect. Such processes with aftereffect will be called composite.

2.3. There exists at least one process with aftereffect determinate throughout the interval $[t_0, T)$ with initial curve determinate in the interval $(\alpha_0, t_0]$.

A process with aftereffect satisfies the axioms of initial data, contraction, articulation, and existence in the time interval $[t_0, T)$, and the initial condition in $(\alpha, t_0]$. Below, we also consider processes with aftereffect $\varphi = \varphi(\varphi_0(\alpha, t_0; t'), t_0, t_1; t)$ not only in the interval $[t_0, t_1)$ but, also, in the interval $[\alpha, t_1)$ and consider such processes to be a composite process with aftereffect consisting of processes

$$t \in (\alpha, t_0), \quad \varphi = \varphi_0(\alpha, t_0; t), \quad t \in [t_0, t), \ \varphi = \varphi(\varphi_0(\alpha, t_0; t'), t_0, t_1; t)$$

We shall write for brevity $\varphi = \varphi(\varphi_0, \alpha, t_0, t_1; t)$, where $(\alpha, t_0]$ indicates the initial distribution interval, and $[t_0, t_1)$ the interval of the process determination. We shall also use the term "process" for a process with aftereffect, with $\varphi \in \Phi$ called the state of the process.

3. The measure $\rho = \rho[\varphi, t]$ of the state of a process with aftereffect at instant of time $t \in [t_0, T)$ is a real number which is brought in correspondence to each pair (φ, t) of any process with aftereffect.

The real number that is brought in correspondence to each curve of initial processes $\varphi_0(\alpha, t_0; t)$, determinate in the interval $(\alpha, t_0]$ is called the measure $\rho_0 = \rho_0[\varphi; \alpha, t_0]$ of initial processes. For instance, $\rho_0[\varphi; \alpha, t_0] = \sup_{t \in (\alpha, t_0]} \rho[\varphi, t]$, where $\rho = \rho[\varphi, t]$ is the measure of the state or processes at any arbitrary instant of time $t \in (\alpha, t_0]$.

If for a given number $\varepsilon > 0$ there exists another number $\delta = \delta(\varepsilon) > 0$ such that the inequality $\rho < \varepsilon$ is satisfied for $\rho_0 < \delta(\varepsilon)$ and every $t \in (\alpha, t_0]$, measure ρ is called upper semicontinuous with respect to measure ρ_0. It is assumed below that measure ρ is upper semicontinuous with respect to ρ_0. The measures ρ and ρ_0 may assume both, positive and negative numerical values, while the numbers ε and $\delta(\varepsilon)$ can only be positive.

4. It is assumed that there exists at least one process with aftereffect $\varphi_* = \varphi_*(\varphi_{0*}(\alpha, t_0; t), t_0, T; t)$. which satisfies the inequalities $\rho_0 \leqslant 0$ and $\rho \leqslant 0$.

The set

$$\Gamma_0 = \{\varphi, t : \rho_0 \leqslant 0, \ t \in (\alpha, t_0]; \ \rho \leqslant 0, \ t \in [t_0, T)\}$$

of processes with aftereffect will be called the set or pipe of unperturbed processes with aftereffect.

Let r be a positve number. We call the set of initial processes that satisfy the inequalities $0 < \rho_0 < r$ the initial perturbations, and the processes issuing from that region will be called perturbed processes with aftereffect.

The unperturbed set Γ_0 of processes with aftereffect is called stable with respect to measures ρ and ρ_0 in the interval $[t_0, T)$, if for any positive number ε it is possible to indicate a positive number $\delta = \delta(\varepsilon)$ such that for any perturbed process with aftereffect $\varphi = \varphi(\varphi_0, \alpha, t_0, T; t)$ which in the initial interval of time $(\alpha, t_0]$ satisfies the inequality $\rho_0 < \delta(\varepsilon)$ throughout the region of process determination, the condition $\rho < \varepsilon$, $\forall t \in [t_0, T)$ is satisfied. If the stability condition is not satisfied, the set Γ_0 is called unstable.

When $T = \infty$ the unperturbed set Γ_0 of processes is called asymptotically stable with respect to the two measures ρ and ρ_0, if it is stable with respect to these measures and provided that the condition $\lim_{t\to\infty} \rho \leqslant 0$ is satisfied by any perturbed processes issuing from the small neighborhood of $\rho_0 \in (0, r)$.

Note that along specific processes the functionals ρ and ρ_0 are assumed to be continuous functions of time t.

5. We introduce the functional $v = v[\varphi, t]$ which at every instant of time $t \in [t_0, T)$ associates to the process state φ the real number v and the functional $v_\alpha = v[\varphi; \alpha, t]$ which for the composite process $\varphi = \varphi(\varphi_0(\alpha, t_0; t''), t_0, T; t')$ associates in the interval $[\alpha, t) \subset [\alpha_0, T)$ at instant of time $t \in [t_0, T)$ the real number ϑ_α. For example

$$v[\varphi, t] = \int_\tau \varphi^2(x, t)\, dx, \quad v_\alpha = v[\varphi; \alpha, t] = \sup_{s \in [\alpha, t]} \vartheta[\varphi, s]$$

where $\varphi = \varphi(x, t)$ is a scalar function of $x \in \tau$ and $t \in [\alpha_0, T)$, and τ is an interval of the real axis.

It is assumed that in unperturbed processes $v = v[\varphi, t] \leqslant 0$ and $v_\alpha = v[\varphi; \alpha, t] \leqslant 0$ when $\rho_0 \leqslant 0$, $t = t_0$, $\rho \leqslant 0$, $t \in [t_0, T)$, and $v[\varphi, t] = 0$ when $\rho[\varphi, t] = 0$.

The functional $v = v[\varphi, t]$ is called uniformly upper semi-continuous with respect to measure ρ_0 in the interval $(\alpha, t_0]$ if for any number $\varepsilon > 0$ can be found a number $\delta = \delta(\varepsilon) > 0$ dependent only on ε and such that the estimate $v < \varepsilon$ is satisfied for $\rho_0 < \delta(\varepsilon)$ and all $t \in (\alpha, t_0]$.

The functional $v_\alpha = v[\varphi; \alpha, t]$ is called upper semi-continuous with respect to measure $\rho_0 = \rho_0[\varphi; \alpha, t]$ at $t = t_0$ if for any arbitrarily small number $\varepsilon > 0$ can be found a positive number $\delta = \delta(\varepsilon)$ such that the estimate $v_\alpha < \varepsilon$ is satisfied under condition that $\rho_0 < \delta(\varepsilon)$, $t = t_0$. If this estimate is satisfied for all $t \in [t_0, T)$, the functional v_α is called uniformly upper semi-continuous with respect to ρ_α in the interval $[t_0, T)$.

The limit of the ratio

$$\lim_{\Delta t \to +0} \frac{v[\varphi_{t+\Delta t}; \alpha(t + \Delta t), t + \Delta t] - v[\varphi_t; \alpha(t), t]}{\Delta t} = \frac{dv}{dt}$$

is called the derivative of functional v_α along the process, where $\varphi_t = \varphi(\varphi_0, \alpha, t_0, T; t)$.

The functional $v = v[\varphi, t]$ is called positive (negative) definite with respect to $\rho = \rho[\varphi, t]$ if $v[\varphi, t] \geqslant 0$ ($v[\varphi, t] \leqslant 0$) when $\rho \geqslant 0$, and $v[\varphi, t] = 0$ when $\rho = 0, t \in [t_0, T)$ and if for any $\varepsilon > 0$ there exists another number $\delta = \delta(\varepsilon) > 0$ such that the inequality $v[\varphi, t] \geqslant \delta(\varepsilon)$ ($v[\varphi, t] \leqslant -\delta(\varepsilon)$) is satisfied for $\rho[\varphi, t] \geqslant \varepsilon$ and all $t \in [t_0, T)$.

The functionals

$$v = \int_0^l f(x)\varphi^2(x, t)\, dx, \quad \infty > c_2 \geqslant f(x) \geqslant c_1 > 0, \qquad v_\alpha = \int_0^l f(x)\varphi^2(x, t)\, dx + \int_{t-\gamma}^{t} \int_0^l \varphi^2(x, t)\, dx\, dt$$

where $\infty > c_2 \geqslant f(x) \geqslant c_1 > 0$ and $\gamma = \text{const} > 0$ are positive definite with respect to measure

$$\rho = \int_0^l \varphi^2(x, t)\, dx$$

Indeed, the estimates $v \geqslant c\rho$, $v_\alpha \geqslant c\rho$, $c > 0$ show that when $\rho \geqslant \varepsilon > 0$ at $t' \in [t - \gamma, t]$, there exists a number $\delta(\varepsilon) = c\varepsilon > 0$ such that $v \geqslant \delta(\varepsilon)$ and $v_\alpha \geqslant \delta(\varepsilon)$. On the other hand, if $\rho = 0$, then $\varphi^2(x, t) = 0$, except the set of measure zero, and consequently $v = 0$ and $v_\alpha = 0$. In this case, the processes that satisfy the equality $\rho = 0$ correspond to an unpertubed process.

In what follows $v = v[\varphi, t]$, $v_\alpha = v[\varphi; \alpha, t]$, $\rho = \rho[\varphi, t]$, $\rho_0 = \rho_0[\varphi; \alpha, t]$ and their derivatives along processes are assumed to be continuous functions of time t in the considered interval.

Note that the functional $v = v[\varphi, t]$ is to be considered as a particular case of functionals of the form $v_\alpha = v[\varphi; \alpha, t]$ with $\alpha = t$, hence it is possible to assume that $v[\varphi, t] = (v_\alpha)_{\alpha=t} = v[\varphi; t, t]$.

6. We present below the theorems on stability and instability.

Theorem 1. For the set Γ_0 of unperturbed processes to be stable with respect to the two measures ρ and ρ_0 it is necessary and sufficient that there exists functional $v_\alpha = v[\varphi; \alpha, t]$ that is positive definite and upper semi-continuous with respect to measures ρ and ρ_0, respectively, when $t \in (\alpha, t_0]$ and non-increasing along perturbed processes with aftereffect.

Theorem 2. For the unperturbed set Γ_0 to be asymptotically stable with respect to the two measures ρ and ρ_0 it is necessary and sufficient that there exists functional $v_\alpha = v[\varphi; \alpha, t]$ upper semi-continuous with respect to measure ρ_0 for $t \in (\alpha, t_0]$ positive definite with respect to measure ρ, nonincreasing along perturbed processes, and satisfying condition $\overline{\lim}\, v_\alpha \leqslant 0$ as $t \to \infty$.

Theorem 3. For the unperturbed set Γ_0 of processes with aftereffect to be unstable with respect to the two measures ρ and ρ_0 it is necessary and sufficient that there exists a bounded functional $v_\alpha = v[\varphi; \alpha, t]$ with positive definite derivative dv_α/dt in region $\{\varphi: v_\alpha > 0\}$, and that there exists for any number $\delta_\alpha > 0$ process $\varphi[\varphi_0, \alpha, t_0; t]$ issuing from region $\{\varphi: v_\alpha > 0\}$ and satisfying the condition $0 < \rho_0 < \delta_0$.

The formulation and proof of these theorems is similar to that of the theorems on stability and instability of process $\varphi \equiv 0$ with respect to the two measures in the absence of aftereffect /4,7/. But the substance of processes and theorems considered here considerably differs from that of processes investigated in /4,7/.

7. Let us consider the theorems on stability of processes with aftereffect, using the derivatives dv/dt and which generalize the results of /1,3/. These theorems define the sufficient stability conditions.

Theorem 4. If for perturbed processes there exists functional $v = v[\varphi, t]$ which in the interval $(\alpha, t_0]$ is upper semi-continuous with respect to measure ρ_0 and positive definite with respect to measure ρ, and whose derivative dv/dt determined at an arbitrary $t \in [t_0, T)$ along perturbed processes on set

$$\{\varphi', t', \varphi, t: v[\varphi', t'] \leqslant v[\varphi, t],\ \alpha \leqslant t_0 \leqslant t' \leqslant t \leqslant T\}$$

is nonpositive for $\alpha \leqslant t_0 \leqslant t' \leqslant t \leqslant T$, the unperturbed set Γ_0 of processes is unstable with respect to measure ρ and ρ_0.

Proof. Functional v is positive definite with respect to measure ρ. Hence for a given number $\varepsilon > 0$ there exists a number $\mu_0 = \mu_0(\varepsilon)$ such that $v \geqslant \mu_0$ if $\rho \geqslant \varepsilon$ and vice versa, $\rho < \varepsilon$ if $v < \mu_0$. Functional v, on the other hand, is upper semi-continuous with respect to measure ρ_0 in the interval $(\alpha, t_0]$. Hence there exists for number $\mu_0 > 0$ a number $\delta_1(\mu_0) > 0$ such that $v < \mu_0$ for $t \in (\alpha, t_0)$, when $\rho_0 < \delta_1(\mu_0)$.

Moreover, the upper semi-continuity of ρ with respect to ρ_0 in the interval $(\alpha, t_0]$ implies that for a given $\varepsilon > 0$ there exists a $\delta_2 = \delta_2(\varepsilon) > 0$ such that $\rho < \varepsilon$ for all $t \in (\alpha, t_0]$ if $\rho_0 < \delta_2(\varepsilon)$. We denote $\delta = \delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$. A number $\delta = \delta(\varepsilon) > 0$ has, thus, been found for $\varepsilon > 0$, such that $v < \mu_0$ and $\rho < \varepsilon$ at all $t \in (\alpha, t_0]$ if $\rho_0 < \delta(\varepsilon)$. By virtue of the assumed continuity of v with respect to t there exists some time interval (t_0, t), where $t > t_0$, $v < \mu_0$, hence $\rho < \varepsilon$.

Let us prove that $\rho < \varepsilon$ for any $t \in (t_0, T)$. Let us assume the existence of a process in which the functional v is differentiable with respect to time t and of the instant of time $t = \tau$ at which $\rho \geqslant \varepsilon$ and $v \geqslant \mu_0$, while up to that instant $v < \mu_0$ and $dv/dt > 0$ at $t = \tau$ in the small neighborhood $(\tau, \tau - \Delta t)$, $\Delta t > 0$. The derivative dv/dt at the instant of time $t = \tau$ depends on the state of the process at $t \leqslant \tau$ in the set

$$\{\varphi', t', \varphi, t: v[\varphi', t'] \leqslant v[\varphi, t] = \mu_0,\ \alpha \leqslant t' < t = \tau < T\}$$

According to the condition of the theorem the condition $dv/dt \leqslant 0$ is satisfied everywhere in the set of such states. Hence $v < \mu_0$ and $\rho < \varepsilon$ at any $t \geqslant t_0$ if $\rho_0 < \delta(\varepsilon)$ at $t = t_0$. The stability of set Γ_0 of processes with aftereffect with respect to measures ρ and ρ_0 is proved.

When proving this theorem it was assumed that T can be finite as well as infinite. Below, in the investigation of asymptotic stability we set $T = \infty$. Note that parameter α may depend on t_0, i.e. $\alpha = \alpha(t_0)$, but $\alpha(t_0) \leqslant t_0$. Duration of the interval of the aftereffect $[\alpha(t_0), t_0]$ of processes considered in the interval $[t_0, \infty)$ depends on the initial instant t_0. The previously proved theorems and the theorem considered below are also valid on these assumptions.

Let us assume that the derivative dv/dt along processes with aftereffect are determined on a set of states φ of a process in some segment $[\beta, t]$. In analyzing its asymptotic stability we shall assume parameter β to be a function of time, i.e. $\beta = \beta(t)$ and $\lim \beta(t) = \infty$ when $t \to \infty$, $\beta(t_0) \leqslant \beta(t) \leqslant t$. The case of $\beta(t) = t$ corresponds to the absence of argument lag. The derivative dv/dt is thus, at instant t a functional determined in the set of states φ on segment $[\beta(t), t]$ whose right- and left-hand ends indefinitely recede to the right from the coordinate origin.

Theorem 5. If there exists for perturbed processes $\{\varphi: 0 < \rho_0 < r\}$ functional $v = v[\varphi, t]$ upper semi-continuous with respect to ρ_0 in interval $(\alpha(t_0), t_0]$ uniformly upper semi-continuous and positve definite with respect to measure ρ for $t \geqslant t_0$ whose derivative dv/dt calculated for an arbitrary $t \in [t_0, \infty)$ along perturbed processes on the set $\{\varphi, t\}$, which satisfy the inequality

$$v[\varphi', t'] \leqslant f(v[\varphi, t]),\ t' \in (\beta(t), t],\ t > t_0, \qquad (v > 0,\ f(v) > v) \tag{7.1}$$

is negative definite with respect to measure ρ for $t \geqslant t_0$, then the set Γ_0 of unperturbed processes with aftereffect is asymptotically stable with respect to measures ρ and ρ_0.

Proof. If the theorem conditions are satisfied, then the conditions of the preceding theorem are satisfied. Hence the set Γ_0 of perturbed processes with aftereffect is stable with respect to the two measures ρ and ρ_0, i.e. there exists for a given number $\varepsilon > 0$ a number $\delta = \delta(\varepsilon) > 0$ such that $\rho < \varepsilon$ for any $t \geqslant t_0$, if $\rho_0 < \delta(\varepsilon)$ at $t = t_0$. It remains to prove the asymptotic stability of set Γ_0.

Let $\varepsilon > 0$. We determine the number $\delta = \delta(\varepsilon)$ such that $\rho < \varepsilon$ for any $t \geqslant t_0$, if $\rho_0 < \delta(\varepsilon)$ at $t = t_0$. We shall consider only such processes for which $\rho_0 < \delta(\varepsilon)$ at $t = t_0$ and, consequently, $\rho < \varepsilon$ for any $t \geqslant t_0$. There exists then a number $\mu_0 = \mu_0(\varepsilon)$ such that

$$t \geqslant t_0, \quad v < \mu_0 \tag{7.2}$$

The functional $v = v[\varphi, t]$ is positive definite with respect to measure ρ, i.e. there exists for any number $\eta \in (0, \varepsilon)$ a number $\mu_T = \mu_T(\eta) > 0$ such that $v \geqslant \mu_T$ when $\rho \geqslant \eta$. This implies that when $v < \mu_T$, then $\rho < \eta$.

Let us ascertain that when we have $v < \mu_0$ at $t = t_0$ the functional v attains the value $v < \mu_T$ in the time interval $[t_0, T]$. To prove this we shall show that there exists a finite time interval $[t_0, T]$ in which the functional decreases by not less than the remainder $\mu_0(\varepsilon) - \mu_T(\eta)$, no matter how small $\mu_T(\eta) > 0$.

Let us assume that $\rho \geqslant \eta$ and, consequently, $v \geqslant \mu_T(\eta)$ at any $t \in [t_0, \infty)$, and show that this assumption is violated in the finite interval $[t_0, T]$.

If $v \geqslant \mu_T(\eta) > 0$, then there exists a positive number $a = a[\mu_T(\eta)] = a(\eta)$ such that $f(v) - v \geqslant a(\eta) > 0$.

The derivative dv/dt in the set of states $v[\varphi', t'] \leqslant f(v[\varphi, t]),\ t' \in [\beta(t), t],\ t_0 \leqslant t$ is a negative definite functional, i.e. there exists for a given number $\eta > 0$ another number $\delta_0 = \delta_0(\eta) > 0$ such that under condition $\rho \geqslant \eta$

$$dv/dt \leqslant -\delta_0 \tag{7.3}$$

Let at the instants of time t and for $\beta = \beta(t)$ the functional $v[\varphi, t]$ be, respectively, equal v_t and v_β. Condition (7.1) postulates the fulfillment of inequality $v_\beta \leqslant f(v_t)$. Since v_β and v_t are not know a priori and depend on the selection of functional v and of processes φ, and are independent of the selection of function $f(v)$, it may happen that at the initial and some other instants within the interval $[\beta(t), t]$, $v_\beta > f(v_t)$ and $v > f(v_t)$. This means that at the instant of time t processes may come from the set of states that are outside the limits of the set $\{\varphi', t', \varphi, t: v \leqslant f(v_t), \beta(t) \leqslant t' \leqslant t\}$, i.e. from the set of states $v > f(v_t), \beta(t) \leqslant t' \leqslant t$. The dependence of dv/dt at instant of time t only on the process states that satisfy condition (7.1) is not guaranteed.

Because of this we shall consider only processes for which the functional $v = v[\varphi, t]$ decreases by less than $a = f(v_t) - v_t$. When v decreases from $v_{t_0} < \mu_0$ to $v_t < \mu_t$ within a finite time interval, the remainder $(\mu_0 - \mu_t)$ proves to be greater than the remainder $(f(v_t) - v_t)$. When the variable v decreases by more than the remainder $f(v_t) - v_t = a$, the estimate (7.2) is inapplicable. Because of this we divide the interval $[\mu_T, \mu_0]$ by $a = a(\eta)$ and introduce the integer N defined by the condition

$$N - 1 \leqslant (\mu_0 - \mu_T)/a \leqslant N \tag{7.4}$$

We shall show that there exist instants of time $t_j = t_j(\eta, \delta, t_0),\ j = 0, 1, \ldots, N$ such that

$$v[\varphi, t] < \mu_T + (N - j)\, a \tag{7.5}$$

for $t > t_j,\ j = 0, 1, \ldots, N$, and condition $v[\varphi, t] < \mu_T$ is consequently satisfied for $t \geqslant T = t_N$.

When $j = 0$ inequality (7.5) is satisfied. Indeed, (7.4) implies that $\mu_T + aN \geqslant \mu_0$. Hence, when $v[\varphi, t] \geqslant \mu_T + aN$ at $t = 0$, then $v[\varphi, t] \geqslant \mu_0$ at $t \geqslant t_0$, which contradicts inequality (7.2), and inequality (7.5) is satisfied when $j = 0$.

Let us assume that inequality (7.5) with subscripts $j = k$ is satisfied at $t \geqslant t_k$ and prove that it is also satisfied at all $j = k + 1$.

It follows from the assumptions that $\beta(t_0) \leqslant \beta(t),\ \beta(t) \leqslant t,\ \lim \beta(t) = \infty$ as $t \to \infty$ that for any given number $t_k > t_0$ and instant of time $t_k^* \geqslant t_k$ such that $\beta(t) \geqslant t_k$ for $t \geqslant t_k^*$ can be found. If $t' \geqslant t_k$, then it follows from (7.5) that $v[\varphi', t'] \leqslant \mu_T + a(N - k)$. We have to ascertain that the inequality $v[\varphi', t'] \leqslant \mu_T - a(N - k - 1)$. is satisfied within a finite time interval. Let us assume that at the instant of time $t > t_k^*$ the inequality $v[\varphi, t_k^*] > \mu_T + a(N - k - 1)$ is also satisfied. Then $\beta(t) \geqslant t_k$ and the process with aftereffect that effects the derivative dv/dt lies inside the polygon

$$\mu_T + a(N-k-1) < v[\varphi', t'] \leqslant \mu_T + a(N-k), \quad \beta(t) \leqslant t' \leqslant t \tag{7.6}$$

Consequently, $dv/dt \leqslant -\delta_0(\eta)$. Assuming the worst case $(v)_{t_k^*} = \mu_T + a(N-k)$ and integrating $dv/dt \leqslant -\delta_0(\eta)$ from $t = t_k^*$ along the process with aftereffect, we obtain

$$v[\varphi, t] \leqslant \mu_T + a(N-k) - (t - t_k^*)\delta_0(\eta)$$

In the time interval $\Delta t_k = a(\eta)/\delta_0(\eta)$ the functional $v[\varphi, t]$ decreases not less than by $a(\eta)$, i.e. $(t - t_k^*)\delta_0(\eta) \geqslant a(\eta)$ at $t = t_{k+1}$, where $t_{k+1} = t_k^* + a(\eta)/\delta_0(\eta)$.

Thus the inequality

$$v[\varphi, t] \leqslant \mu_T + a(N-k-1)$$

is satisfied for $t \geqslant t_{k+1}$.

Its violation at some $t \geqslant t_{k+1}$ contradicts the negative definiteness of the derivative dv/dt in region (7.6). Thus, beginning from some instant of time $t = t_{k+1}$, inequality (7.5) with subscript $j = k+1$ is satisfied. Applying the method of complete induction, we find that inequality (7.5) is satisfied for any j, including $j = N$. Setting in (7.5) $j = N$ we obtain

$$t \geqslant t_N = T. \quad v[\varphi, t] < \mu_T$$

Hence the assumption that $v > \mu_T$ is satisfied at any $t \geqslant t_0$ is violated at finite $T = t_N$. When $t \geqslant t_N = T$, $v < \mu_T$ is satisfied and, consequently $\rho < \eta$.

Thus within a finite time interval ρ remains smaller than any arbitrarily small positive number η, i.e. $\overline{\lim}\, \rho \leqslant 0$ as $t \to 0$. Q.E.D.

REFERENCES

1. RAZUMIKHIN B.S., On the stability of systems with lag. PMM, Vol.20, No.4, 1956.
2. KRASOVSKII N.N., Certain Problems of the Theory of Motion Stability. Moscow, Fizmatgiz, 1959. (See also Stanford Univ. Press, Stanford Cal. 1963).
3. SIRAZETDINOV T.K. and SEMENOV P.K., On the stability of processes with distributed parameters and lag. In: Problems of Analytical Mekhaniks, Theory of Stability of Motion and Control. Moscow, "Nauka", 1975.
4. MOVCHAN A.A., Stability of processes with respect to two metrics. PMM, Vol.24, No.6, 1960.
5. MATROSOV V.M. and ANAPOL'SKII L.Iu., The method of matching in process analysis. In: Problems of Analytical Mechanics, Theory of Stability of Motion and Control. Moscow, "Nauka", 1975.
6. SIRAZETDINOV T.K., Stability process sets. In: Problems of Motion Stability, Analytical Mechanics, and Motion Control. "Nauka", SO Akad. Nauk SSSR, 1979.
7. SIRAZETDINOV T.K., Stability of Systems with Distributed Parameters. Izd. Kazansk. Aviats. Inst., 1971.

Translated by J.J.D.